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Matrix Calculations for Liapunov Quadratic Forms

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1. INTRODUCTION

The first half of this paper concerns the linear system of ordinary differential equations

$$dx/dt = Ax, \tag{1.1}$$

where A is a constant $n \times n$ matrix with complex elements. An asterisk will denote the conjugate transpose of a matrix. That is, $A^* = (\bar{a}_{ji})$ when $A = (a_{ij})$. If $L(x) = x^*Px$, where P is a constant hermitian matrix, then the derivative of $L(x)$ with respect to t, following a solution $x(t)$ of (1.1), is $dL/dt = -2x^*Qx$ with

$$-2Q = A^*P + PA. \tag{1.2}$$

If P, Q are positive definite matrices then the quadratic form $L(x)$ is a Liapunov function for (1.1). When A and hermitian Q are given, it is of interest to find an hermitian matrix P satisfying (1.2). This problem arises, for example, in discussing the stability of nonlinear control systems (see [*1*], p. 85). Bedel'baev ([*2*], p. 38) describes four main methods for the practical computation of $L(x)$, due, respectively, to Liapunov, Moisseev, Bedel'baev and Lur'e. Another method is given by Tsai [*3*]. The methods of Liapunov and Moisseev require the inversion of a matrix of order $\frac{1}{2}n(n+1)$. Those of Bedel'baev and Tsai require the expansion of many complicated determinants. That of Lur'e involves the computation of the eigenvalues of A and the reduction of (1.1) to a canonical form. All these methods require heavy computation when $n > 3$. Part of their complexity is caused by the large number of variables arising from the n^2 elements of A. In Section 2 of the present paper, an explicit expression for the matrix P is obtained, which uses the n^2 elements of A in the relatively simple operations of matrix multiplication and addition. But in the more difficult operations of determinantal expansion, only n variables $k_1, \cdots, k_n$ are involved.

In the second half of the paper, the analogous problem is considered for difference equations of the form

$$x(t+1) = Ax(t). \tag{1.3}$$

To discuss nonlinear perturbations of (1.3), Hahn [4] has used a Liapunov function of the form $L(x) = x^*Rx$, with

$$A^*RA - R = -Q, \tag{1.4}$$

where R, Q are positive definite hermitian matrices. In Section 5 of the present paper, it is shown how to find an hermitian matrix R satisfying (1.4) when A and hermitian Q are given.

2. Solution of (1.2)

Define constants $k_1, \cdots, k_n$ and an $n \times n$ matrix G by

$$\det(\lambda I - A) = \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_n, \tag{2.1}$$

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -k_n & -k_{n-1} & -k_{n-2} & -k_{n-3} & \cdots & -k_1 \end{pmatrix}. \tag{2.2}$$

Theorem 1. *If T is any $n \times n$ matrix and $U = (u_{ij})$, $C = (c_{ij})$ are $n \times n$ matrices satisfying*

$$G^*U + UG = -2C, \tag{2.3}$$

then the following matrices satisfy (1.2):

$$P = \Sigma_i\Sigma_j u_{ij}(A^*)^{i-1}TA^{j-1}, \qquad Q = \Sigma_i\Sigma_j c_{ij}(A^*)^{i-1}TA^{j-1},$$

where Σ_i denotes summation over $i = 1, 2, \cdots, n$.

Proof. If $G = (g_{ij})$ then (2.2) shows that

$$\Sigma_j g_{rj}A^{j-1} = A^r, \tag{2.4}$$

for $r = 1, 2, \cdots, n-1$. Since A satisfies its own characteristic equation, (2.2) also gives

$$\Sigma_j g_{nj}A^{j-1} = -\Sigma_j k_{n+1-j}A^{j-1} = A^n.$$

Hence, (2.4) also holds with $n = r$. From (2.3), we get

$$\Sigma_r(\bar{g}_{ri}u_{rj} + u_{ir}g_{rj}) = -2c_{ij}. \tag{2.5}$$

If P is as stated in Theorem 1, then (2.4) gives

$$PA = \Sigma_i \Sigma_r u_{ir} (A^*)^{i-1} TA^r = \Sigma_i \Sigma_r \Sigma_j u_{ir} g_{rj} (A^*)^{i-1} TA^{j-1},$$
$$A^*P = \Sigma_r \Sigma_j u_{rj} (A^*)^r TA^{j-1} = \Sigma_i \Sigma_r \Sigma_j u_{rj} \bar{g}_{ri} (A^*)^{i-1} TA^{j-1}.$$

Now add these together and use (2.5):

$$A^*P + PA = \Sigma_i \Sigma_j (-2c_{ij}) (A^*)^{i-1} TA^{j-1} = -2Q.$$

This establishes Theorem 1. The following is a special case of it:

COROLLARY. *If $U = (\hat{u}_{ij})$ is a solution of (2.3) when $C = \operatorname{diag}(c, 0, \cdots, 0)$, then*

$$P = \Sigma_i \Sigma_j \hat{u}_{ij} (A^*)^{i-1} TA^{j-1} \tag{2.6}$$

is a solution of

$$A^*P + PA = -2cT. \tag{2.7}$$

If $c \neq 0$, then (2.6) with $T = c^{-1}Q$ is a solution of (1.2). Furthermore, this solution P is obviously hermitian when T and $(\hat{u}_{ij})$ are hermitian. Usually, c is taken to be the Hurwitz determinant h defined by

$$h = \begin{vmatrix} k_1 & k_3 & k_5 & k_7 & \cdots & k_{2n-1} \\ k_0 & k_2 & k_4 & k_6 & \cdots & k_{2n-2} \\ 0 & k_1 & k_3 & k_5 & \cdots & k_{2n-3} \\ 0 & k_0 & k_2 & k_4 & \cdots & k_{2n-4} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdot & \cdots & k_n \end{vmatrix},$$

where $k_0 = 1$ and $k_\nu = 0$ for all $\nu > n$. This determinant occurs in the Routh-Hurwitz criteria for the stability of A (see [5], p. 406).

THEOREM 2. *If $k_1, \cdots, k_n$ are real, then the equation*

$$G^*V + VG = -2 \operatorname{diag}(h, 0, \cdots, 0) \tag{2.8}$$

has an hermitian solution $V = (v_{\alpha\beta})$ given by

$$v_{\alpha\beta} = (-1)^{\alpha-1} \sum_{r=0}^{n-\alpha} \sum_{s=0}^{n-\beta} (-1)^r k_r k_s \phi(\tfrac{1}{2}[\alpha + \beta + r + s]), \tag{2.9}$$

where $k_0 = 1$, $\phi(\nu) = 0$ when ν is not an integer, and otherwise $\phi(\nu)$ is the cofactor of the element in the first row and ν-th column of the determinant h.

Proof. It is easy to verify from (2.2) that

$$\det(\lambda I - G) = \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_n .$$

Assume, for the present, that $k_1, \cdots, k_n$ satisfy the Routh-Hurwitz inequalities, so that all the eigenvalues of G have negative real parts. It is proved elsewhere (see [*6*], Theorem 1) that when this is so, a solution of (2.3) is

$$U = \pi^{-1} \int_{-\infty}^{+\infty} (G^* + iyI)^{-1} C(G - iyI)^{-1} \, dy, \tag{2.10}$$

and that this is hermitian when C is hermitian. Now,

$$(\lambda I - G)^{-1} = \Delta(\lambda)^{-1} (\Delta_{\alpha\beta}(\lambda)),$$

where $\Delta(\lambda) = \det(\lambda I - G)$ and $\Delta_{\alpha\beta}(\lambda)$ is the cofactor of the element in the α-th column and β-th row of $\Delta(\lambda)$. If $C = \operatorname{diag}(h, 0, \cdots, 0)$ then $U = (v_{\alpha\beta})$ and (2.10) gives

$$v_{\alpha\beta} = \pi^{-1} h \int_{-\infty}^{+\infty} \Delta(iy)^{-1} \Delta(-iy)^{-1} \Delta_{1\alpha}(-iy) \Delta_{1\beta}(iy) \, dy, \tag{2.11}$$

provided that $k_1, \cdots, k_n$ are real. The cofactor $\Delta_{1\alpha}(\lambda)$ of the element in the first column and α-th row of $\det(\lambda I - G)$ can be found from (2.2). It is

$$\Delta_{1\alpha}(\lambda) = \sum_{r=0}^{n-\alpha} k_r \lambda^{n-\alpha-r}, \tag{2.12}$$

where $k_0 = 1$. Substituting this in (2.11), we get

$$v_{\alpha\beta} = 2h \sum_{r=0}^{n-\alpha} \sum_{s=0}^{n-\beta} (-1)^{n-\alpha-r} k_r k_s \Gamma_{2n-\alpha-\beta-r-s} , \tag{2.13}$$

where

$$\Gamma_\nu = (2\pi)^{-1} \int_{-\infty}^{+\infty} \Delta(iy)^{-1} \Delta(-iy)^{-1} (iy)^\nu \, dy.$$

When $\nu = 1, 3, 5, \cdots, 2n - 3$, the integrand is an odd function of y and therefore $\Gamma_\nu = 0$. Also, Bedel'baev ([*7*], p. 27) has proved that

$$\Gamma_{2\sigma} = \tfrac{1}{2}(-1)^{n-1} h^{-1} \phi(n - \sigma),$$

for $\sigma = 0, 1, 2, \cdots, n - 1$, where $\phi(\nu)$ is defined in the statement of Theorem 2. Hence, (2.13) reduces to (2.9). It has now been proved that (2.9) satisfies (2.8) provided that $k_1, \cdots, k_n$ satisfy the Routh-Hurwitz inequalities. To remove this assumption, observe that when (2.9) is substituted in (2.8), both sides are polynomials in $k_1, \cdots, k_n$. This finishes the proof of Theorem 2.

3. Some Practical Details

If $k_1, \cdots, k_n$ in (2.1) are real and $h \neq 0$ then an hermitian solution P of (1.2) is obtained by putting $\hat{u}_{ij} = v_{ij}$ and $T = h^{-1}Q$ in (2.6). The numbers $k_1, \cdots, k_n$ can be found, without determinantal expansion, by the following method explained in ([8], p. 279). Since $k_1, \cdots, k_n$ are homogeneous polynomials in the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A, they can be expressed by Newton's formulae in terms of the homogeneous polynomials $s_1, s_2, \cdots, s_n$ and then s_ν can be computed from the formula

$$s_\nu = \lambda_1{}^\nu + \cdots + \lambda_n{}^\nu = \text{trace } A^\nu.$$

This method is particularly convenient because the computation of A^ν is already required for (2.6). The cofactors $\phi(\nu)$ in (2.9) are relatively easy to evaluate because h involves only the variables $k_1, \cdots, k_n$ and many of its elements are zero. In the following tables, $(v_{\alpha\beta})$ and h are given in the cases $n = 2, 3, 4$:

Case $n = 2$.

$$v_{22} = 1, \qquad v_{11} = k_2 + k_1{}^2,$$
$$v_{12} = v_{21} = k_1, \qquad h = k_1 k_2.$$

Case $n = 3$.

$$v_{33} = k_1, \qquad v_{22} = k_3 + k_1{}^3,$$
$$v_{11} = k_3 k_1{}^2 + k_1 k_2{}^2 - k_2 k_3, \qquad v_{32} = v_{23} = k_1{}^2,$$
$$v_{31} = v_{13} = k_1 k_2 - k_3, \qquad v_{21} = v_{12} = k_1{}^2 k_2,$$
$$h = k_3(k_1 k_2 - k_3).$$

Case $n = 4$.

$$\begin{aligned}
v_{44} &= k_1 k_2 - k_3, \\
v_{33} &= (k_1 k_2 - k_3)\, k_1{}^2 + k_1 k_4, \\
v_{22} &= (k_1 k_2 - k_3)(k_2{}^2 - 2k_4) + k_4(k_1{}^3 - k_3), \\
v_{11} &= (k_1 k_2 - k_3)(k_3{}^2 + k_2 k_4) - k_1{}^2 k_3 k_4 - k_1 k_4{}^2, \\
v_{43} = v_{34} &= (k_1 k_2 - k_3)\, k_1, \\
v_{42} = v_{24} &= (k_1 k_2 - k_3)\, k_2 - k_1 k_4, \\
v_{41} = v_{14} &= (k_1 k_2 - k_3)\, k_3 - k_1{}^2 k_4, \\
v_{32} = v_{23} &= (k_1 k_2 - k_3)\, k_1 k_2, \\
v_{31} = v_{13} &= (k_1 k_2 - k_3)(k_1 k_3 + k_4) - k_1{}^3 k_4, \\
v_{21} = v_{12} &= (k_1 k_2 - k_3)\, k_2 k_3 - k_1 k_3 k_4, \\
h &= k_4\{k_3(k_1 k_2 - k_3) - k_1{}^2 k_4\}.
\end{aligned}$$

4. Singular Cases

If $h = 0$, then the matrix P got from (2.6) by putting $\hat{u}_{ij} = v_{ij}$, satisfies $A^*P + PA = 0$. In this case, (1.1) has the quadratic integral $x^*Px = \text{constant}$, provided that $P \neq 0$. But for certain values of $k_1, \cdots, k_n$ it can happen that (2.9) gives $V = 0$ and then (2.6) produces the trivial solution $P = 0$. Differentiation of (2.8) gives

$$(d_\alpha G^*) V + G^* d_\alpha V + (d_\alpha V) G + V d_\alpha G = -2 \operatorname{diag}(d_\alpha h, \cdots, 0),$$

where d_α is the operator $\partial/\partial k_\alpha$. If $V = 0$, this shows that (2.3) is satisfied by $U = d_\alpha V$, $C = \operatorname{diag}(d_\alpha h, \cdots, 0)$. Then (2.6) with $\hat{u}_{ij} = d_\alpha v_{ij}$ is a solution of (2.7) with $c = d_\alpha h$. But if it also happens that $d_\alpha V = 0$ for $\alpha = 1, \cdots, n$, then a second differentiation of (2.8) shows that (2.6) with $u_{ij} = d_\beta d_\alpha v_{ij}$ is a solution of (2.7) with $c = d_\beta d_\alpha h$. By differentiating (2.8) a sufficient number of times we can always obtain a nontrivial equation.

5. Difference Equations

The solving of (1.4) is closely connected with the solving of (1.2) as the following theorem shows. Let

$$B = (A + Ie^{i\nu})(A - Ie^{i\nu})^{-1}, \tag{5.1}$$

where ν is real and $e^{i\nu}$ is not an eigenvalue of A.

Theorem 3. *R satisfies (1.4) if and only if*

$$P = (A^* - Ie^{-i\nu}) R(A - Ie^{i\nu}) \tag{5.2}$$

satisfies

$$B^*P + PB = -2Q.$$

Proof. Substitute (5.2) into the right-hand side of the identity

$$2(B^*P + PB) = (B^* + I) P(B + I) - (B^* - I) P(B - I)$$

and then use the relations

$$(A - Ie^{i\nu})(B + I) = 2A, \qquad (A - Ie^{i\nu})(B - I) = 2Ie^{i\nu},$$

which are obtained from (5.1). The result is

$$2(B^*P + PB) = 4(A^*RA - R),$$

from which Theorem 3 follows at once.

If λ is an eigenvalue of A then $\mu = (\lambda + e^{i\nu})(\lambda - e^{i\nu})^{-1}$ is an eigenvalue of B and $\operatorname{Re}\mu < 0$ if and only if $|\lambda| < 1$. This remark shows that the following theorem is related to a result obtained elsewhere (see [6], Theorem 1).

THEOREM 4. *If $|\lambda| < 1$ for every eigenvalue λ of A, then (1.4) is satisfied by*

$$R = (2\pi i)^{-1} \oint (A^* - Iz^{-1})^{-1} Q (A - Iz)^{-1} z^{-1} dz, \tag{5.3}$$

where $\oint$ denotes anticlockwise integration round the circle $|z| = 1$ in the complex z plane.

Proof. Using the identities

$$(A - Iz)^{-1} A = I + z(A - Iz)^{-1},$$
$$A^*(A^* - Iz^{-1})^{-1} = I + z^{-1}(A^* - Iz^{-1})^{-1},$$

we deduce from (5.3) that

$$A^*RA = Q + R + (2\pi i)^{-1}[J_1 Q + Q J_2], \tag{5.4}$$

where

$$J_1 = \oint (A^* z - I)^{-1} z^{-1} dz, \qquad J_2 = \oint (A - Iz)^{-1} dz.$$

Since the eigenvalues of A^* all lie inside unit circle, the integrand of J_1 is regular on and inside the path of integration, except for a simple pole at $z = 0$ with residue $-I$. Hence, $J_1 = -2\pi i I$. If the variable z in J_2 is changed to z^{-1}, it becomes

$$J_2 = \oint (Az - I)^{-1} z^{-1} dz.$$

The same argument now gives $J_2 = -2\pi i I$. With J_1, J_2 replaced by $-2\pi i I$, (5.4) reduces to (1.4). This establishes Theorem 4.

THEOREM 5. *If T is any $n \times n$ matrix and $U = (u_{ij})$, $C = (c_{ij})$ are $n \times n$ matrices satisfying*

$$G^*UG - U = -C, \tag{5.5}$$

then the following matrices satisfy (1.4):

$$R = \Sigma_i \Sigma_j u_{ij} (A^*)^{i-1} T A^{j-1}, \qquad Q = \Sigma_i \Sigma_j c_{ij} (A^*)^{i-1} T A^{j-1}.$$

Proof. With $G = (g_{ij})$, (5.5) can be written as

$$\Sigma_r \Sigma_s \bar{g}_{ri} u_{rs} g_{sj} = u_{ij} - c_{ij}.$$

From this and (2.4), it follows that

$$\begin{aligned} A^*RA &= \Sigma_r \Sigma_s u_{rs} (A^*)^r\, TA^s, \\ &= \Sigma_r \Sigma_s \Sigma_i \Sigma_j \bar{g}_{ri} u_{rs} g_{sj} (A^*)^{i-1}\, TA^{j-1}, \\ &= \Sigma_i \Sigma_j (u_{ij} - c_{ij})\, (A^*)^{i-1}\, TA^{j-1}, \\ &= R - Q. \end{aligned}$$

This proves Theorem 5. The following is a special case of it:

COROLLARY. *If* $U = (\hat{u}_{ij})$ *is a solution (5.5) when* $C = \operatorname{diag}(c, 0, \cdots, 0)$, *then*

$$R = \Sigma_i \Sigma_j u_{ij} (A^*)^{i-1}\, TA^{j-1} \tag{5.6}$$

is a solution of $A^*RA - R = -cT$.

If $c \neq 0$, then a solution of (1.4) is obtained by putting $T = c^{-1}Q$ in (5.6). Furthermore, this solution is obviously hermitian when T and $(\hat{u}_{ij})$ are hermitian. Usually, c is taken to be the determinant f of order $(n+1)$ defined by

$$f = \begin{vmatrix} k_0 & k_1 + k_{-1} & k_2 + k_{-2} & \cdots & k_n + k_{-n} \\ k_1 & k_2 + k_0 & k_3 + k_{-1} & \cdots & k_{1+n} + k_{1-n} \\ k_2 & k_3 + k_1 & k_4 + k_0 & \cdots & k_{2+n} + k_{2-n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ k_n & k_{n+1} + k_{n-1} & k_{n+2} + k_{n-2} & \cdots & k_{2n} + k_0 \end{vmatrix}$$

where $k_0 = 1$, $k_\nu = 0$ for all $\nu < 0$ and $k_\nu = 0$ for all $\nu > n$.

THEOREM 6. *If* $k_1, \cdots, k_n$ *are real, then the equation*

$$G^*WG - W = -\operatorname{diag}(f, 0, \cdots, 0) \tag{5.7}$$

has an hermitian solution $W = (w_{\alpha\beta})$ *given by*

$$w_{\alpha\beta} = \sum_{r=0}^{n-\alpha} \sum_{s=0}^{n-\beta} k_r k_s \psi(|\,\alpha + r - \beta - s\,|), \tag{5.8}$$

where $k_0 = 1$ *and* $\psi(\nu)$ *is the cofactor of the element in the first row and* $(\nu + 1)$-th *column of the determinant* f.

The proof of Theorem 6 uses the following lemma.

LEMMA. *If* $\Delta(z) = \det(zI - G)$ *has no zeros outside the disc* $|z| < 1$ *and*

$$\Omega_\nu = (2\pi i)^{-1} \oint \Delta(z^{-1})^{-1}\, \Delta(z)^{-1}\, z^{\nu-1} dz, \tag{5.9}$$

where $\oint$ *is taken anticlockwise round the circle* $|z| = 1$, *then*

$$\Omega_{-\nu} = \Omega_\nu = f^{-1}\psi(\nu), \qquad \textit{for} \qquad \nu = 0, 1, \cdots, n.$$

Proof of lemma. The relation $\Omega_{-\nu} = \Omega_\nu$ follows at once by changing the variable z in (5.9) to z^{-1}. Since $k_r = 0$, except for $r = 0, 1, \cdots, n$, it follows that

$$\sum_{\nu=-n}^{n} k_{\nu+s}\Omega_{-\nu} = \sum_{r=0}^{n} k_r\Omega_{s-r},$$

$$= (2\pi i)^{-1} \oint \Delta(z^{-1})^{-1} \Delta(z)^{-1} \sum_{r=0}^{n} k_r z^{s-r-1} dz,$$

$$= (2\pi i)^{-1} \oint \Delta(z^{-1})^{-1} z^{s-n-1} dz,$$

for $s = 0, 1, \cdots, n$. Since the polynomial $z^n\Delta(z^{-1})$ has no zeros in the disc $|z| \leqslant 1$, the integrand is regular in $|z| \leqslant 1$ when $s \geqslant 1$. But when $s = 0$, it has a pole of order 1 and residue 1 at $z = 0$. The value of the integral is therefore $2\pi i\delta_{0s}$, where δ_{0s} is Kronecker's symbol. Since $\Omega_{-\nu} = \Omega_\nu$, the equation can be written as

$$k_s\Omega_0 + \sum_{\nu=1}^{n} (k_{s+\nu} + k_{s-\nu})\,\Omega_\nu = \delta_{0s},$$

for $s = 0, 1, \cdots, n$. This is a system of $(n + 1)$ linear equations for the unknowns $\Omega_0, \Omega_1, \cdots, \Omega_n$. Solving by Cramer's method, we get $\Omega_\nu = f^{-1}\psi(\nu)$.

Proof of Theorem 6. If it is assumed that $|\lambda| < 1$ for every eigenvalue λ of G, then Theorem 4 shows that (5.5) has the solution

$$U = (2\pi i)^{-1} \oint (G^* - Iz^{-1})^{-1}\, C(G - Iz)^{-1}\, z^{-1} dz.$$

Since $U = (w_{\alpha\beta})$ when $C = \operatorname{diag}(f, 0, \cdots, 0)$, this gives

$$w_{\alpha\beta} = (2\pi i)^{-1} f \oint \Delta(z^{-1})^{-1} \Delta(z)^{-1} \Delta_{1\alpha}(z^{-1})\, \Delta_{1\beta}(z)\, z^{-1} dz,$$

where $\Delta_{\alpha\beta}(z)$ is as in (2.11). Substituting (2.12), we get

$$w_{\alpha\beta} = f \sum_{r=0}^{n-\alpha} \sum_{s=0}^{n-\beta} k_r k_s \Omega_{\alpha+r-\beta-s},$$

where Ω_ν is defined by (5.9). From this, (5.8) follows by use of the lemma.

Finally, the assumption that $|\lambda| < 1$ for every eigenvalue λ of G, can be deleted since both sides of (5.7) are polynomials in $k_1, \cdots, k_n$. This establishes Theorem 6.

The following tables give $(w_{\alpha\beta})$ and f in the case $n = 2, 3$:

CASE $n = 2$.

$$
\begin{aligned}
w_{22} &= 1 + k_2, \\
w_{11} &= 1 + k_2 - k_1^2 + k_1^2 k_2, \\
w_{21} &= w_{12} = k_1 k_2, \\
f &= 1 + k_2 - k_1^2 + k_1^2 k_2 - k_2^2 - k_2^3.
\end{aligned}
$$

CASE $n = 3$.

$$
\begin{aligned}
w_{33} &= 1 + k_2 - k_1 k_3 - k_3^2, \\
w_{22} &= (1 + k_1^2)(1 + k_2 - k_1 k_3 - k_3^2) + 2k_1(k_2 k_3 - k_1), \\
w_{11} &= (1 + k_1^2 - k_2^2)(1 + k_2 - k_1 k_3 - k_3^2) - 2(k_1 - k_2 k_3)^2, \\
w_{32} &= w_{23} = (k_2 - k_1 k_3)(k_1 + k_3), \\
w_{31} &= w_{13} = k_3(k_1 - k_2 k_3), \\
w_{21} &= w_{12} = k_3(k_2 - k_1 k_3)(1 + k_2), \\
f &= (1 + k_1^2 - k_2^2 - k_3^2)(1 + k_2 - k_1 k_3 - k_3^2) - 2(k_1 - k_2 k_3)^2.
\end{aligned}
$$

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